Parallel axis theorem / Steiner's theorem

Related topics

Rigid body, moment of inertia, centre of gravity, axis of rotation, torsional vibration, spring constant, angular restoring force.

Principle and task

The period of vibration of a circular disc which performs torsional vibrations about various parallel axes, is measured. The moment of inertia of the disc is determined as a function of the perpendicular distance of the axis of rotation from the centre of gravity.

Equipment

Rotation axle	02415.01	1
Disk, w. diametrical holes	02415.07	1
Spring balance, transparent, 2 N	03065.03	1
Light barrier with Counter	11207.08	1
Power supply 5 V DC/0.3 A	11076.93	1
Tripod base -PASS-	02002.55	1
Barrel base -PASS-	02006.55	1
Rule, plastic, 200 mm	09937.01	1

Problems

1. Determination of the angular restoring constant of the spiral spring.
2. Determination of the moment of inertia of a circular disc as a function of the perpendicular distance of the axis of rotation from the centre of gravity.

Set-up and procedure

The experimental set-up is arranged as shown in Fig. 1. In order to measure the angular restoring factor, the disc is fixed on the torsion axis at its centre of gravity. With the spring balance, which acts in a hole in the disc, the force needed to deflect the disc through a given angle is measured. When doing this, the lever arm forms a right angle with the spring balance. It is convenient to select an angle of 180°, since the row of holes can thus be used as a "protractor".

For measuring the vibration period of the disc, a mask (width ≤ 3 mm) is stuck on, on the line of the row of holes. The light barrier is pushed over this mask with the disc in its position of rest.

Switch the light barrier to -mode.

Fig. 1: Experimental set-up for measuring the moment of inertia (Steiner's theorem).

The disc is deflected through about 180° and the half-cycle time of the vibration is measured with the counter, anticlockwise and clockwise measurements being averaged. For safety and stability reasons, it is recommended that the spring should not be twisted beyond ± 720°.

Theory and evaluation

The relationship between the angular momentum $\vec{L}$ of a rigid body in a stationary coordinate system with its origin at the centre of gravity, and the moment $\vec{T}$ acting on it, is

$$\vec{T} = \frac{d}{dt} \vec{L}. \qquad (1)$$

The angular momentum is expressed by the angular velocity $\vec{\omega}$ and the inertia tensor $\hat{I}$ from

$$\vec{L} = \hat{I} \odot \vec{\omega},$$

that is, the reduction of the tensor with the vector.

In the present case, $\vec{\omega}$ has the direction of a principal inertia axis (z-axis), so that $\vec{L}$ has only one component:

$$L_Z = I_Z \cdot \omega,$$

where I_Z is the z-component of the principal inertia tensor of the plate. For this case, equation (1) reads:

$$T_Z = I_Z \frac{d\omega}{dt} = I_Z \frac{d^2\phi}{dt}.$$

where ϕ is the angle of rotation.

In the Hooke's law range, the moment of a spiral spring is:

$$T_Z = -D \cdot \phi \qquad (2)$$

where D is the angular restoring constant.

From the regression line to the measured values of Fig. 2 with the linear statement

$$Y = A + BX$$

the slope

$$B = 0.0255 \text{ Nm/rad}. \qquad \text{(see (2))}$$

The angular restoring factor, from (2) is

$$D = 0.0255 \text{ Nm/rad}.$$

The equation of motion reads:

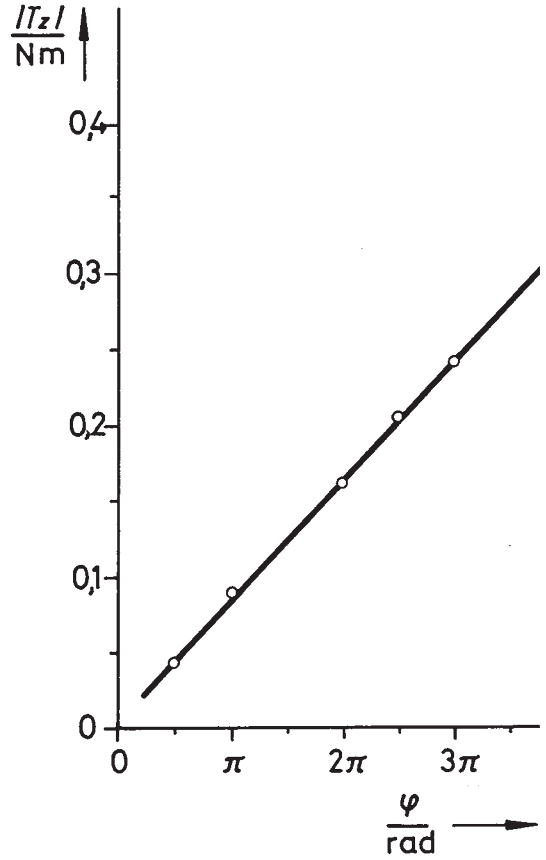
$$\frac{d^2\phi}{dt^2} + \frac{D}{I_Z} \phi = 0.$$

The period and frequency of this vibration are:

$$T = 2\pi \sqrt{I_Z/D}$$

$$f = \frac{l}{2\pi} \sqrt{D/I_Z}.$$

Fig. 2: Moment (torque) of a spiral spring as a function of the angle of rotation.



If ρ (x, y, z) is the density distribution of the body, the moment of inertia I_Z is obtained as

$$I_Z = \iiint (x^2 + y^2)\, \rho\, (x, y, z)\, dx\, dy\, dz.$$

The origin of coordinates is located at the centre of gravity. The same component of the inertia tensor, in relation to a coordinate origin displaced by a, is

$$I_Z' = I_Z m a^2,$$

where m is the mass of the body.
Therefore

$$T^2 = \frac{4\pi^2}{D} \cdot (I_Z + m a^2) \qquad (3)$$

From the regression line to the measured values of Fig. 3 and the linear statement

$$Y = A + BX$$

one obtains

$$A = 6.86 \text{ s}^2 \pm 0.15 \text{ s}^2 \qquad \text{(see (3))}$$

and from this the moment. of inertia of the disc with the axis of rotation through the centre of gravity

$$I_Z = 4.52 \cdot 10^{-3} \text{ kgm}^2.$$

Fig. 3: Vibration period of a disc as a function of the perpendicular distance of the axis of rotation from the centre of gravity.