Lecture Notes

Mechanism Design and Analysis
(MDA)

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Preliminary Remarks

Mechanism design and analysis (MDA) is one of the most prominent subject of mechanical and mechatronics engineering. It is also the logical sequel to the lectures „Technische Mechanik“ in that it will now be dealt with multiple bodies in planar and spatial motion. In past and future engineers are involved in the development of sophisticated mechanisms.

Content overview

Introduction of mechanism design:
- modelling by rigid bodies and joints,
- discussion of topology as tree structures and closed loops,
- state variables and degrees of freedom (DOFs) of joints and system,
- transfer functions

Design of simple planar mechanisms, Introduction into parameter optimization
- slider crank, four-bar-mechanism

Kinematical analysis
- frames and orientation matrix,
- functions of position, velocity and acceleration,
- discussion of mechanism behaviour,
- graphical methods

Dynamical analysis
- equilibrium conditions,
- principle of virtual power,

Introduction to multibody programs
- demonstrations on examples

Goals and Objectives

Students will be able to
- understand the movement of mechanisms and to calculate the DOFs of a system
- setup the kinematical transfer functions of a planar mechanism
- calculate the applied forces and torques of the input links.

Prerequisites

Courses as Technical Mechanics I and II, Mathematics I and II, Signals and Systems, (Modelling and Simulation)
Notations

1. General variables

**Scalars** arbitrary letters including Greek letters, e.g. \(a, b, P, x_i, \alpha, \beta, \gamma, \lambda\)

**Indices** with letters in lower case, e.g. \(i, j, k, l\)

**Matrices and vectors** are lists of scalars. A vector is a column of a matrix.

**Vectors** are denoted by letters in lower case, for the manuscript in bold face, e.g. \(x = (x_i), \quad i = 1, 2, 3, \ldots, n\)

for hand writing the letter is underlined, e.g. \(\underline{x} = (\underline{x_i})\)

**Norm** \(\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}\)

**Matrices** are denoted by capital letters, for the manuscript in bold face, e.g. \(M = (M_{ij}), \quad i = 1, 2, 3, \ldots, n; \quad j = 1, 2, 3, \ldots, m\)

for hand writing the letter is double underlined, e.g. \(\underline{M} = (\underline{M}_{ij})\)

2. “Physical Vector” in space \(\mathbb{R}^2\) or \(\mathbb{R}^3\)

A vector is an invariant of coordinate systems

**Vectors** denoted by arbitrary letters and marked by a arrow at the head, e.g.

\(\vec{v}, \vec{F}\)

**Absolute value** or length or amount of the vector, e.g.

\(v = |\vec{v}|; \quad F = |\vec{F}|\)

3. Representation of a vector in a coordinate system (frame)

with basis vectors \(\vec{e}_1, \vec{e}_2, \vec{e}_3\) (3D or 2D),

where \(|\vec{e}_1| = 1\), e.g.

\[\vec{v} = \vec{e}_1v_1 + \vec{e}_2v_2 + \vec{e}_3v_3 \equiv \vec{e}^T v = v^T \vec{e}\]

where \(v = (v_i) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \vec{e} = (\vec{e}_i) = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}\)

and \(v_1, v_2, v_3\) are coordinates or components of vector \(\vec{v}\).

Especially: **Cartesian right-hand frame**

\(\vec{e}_i \cdot \vec{e}_j = \delta_{ij}\) leads \(\vec{e} \cdot \vec{e}^T = E\)

\(\vec{e}_i \times \vec{e}_j = \varepsilon_{ijk} \vec{e}_k\) leads \(\varepsilon \times \varepsilon^T = \begin{pmatrix} 0 & -\vec{e}_3 & \vec{e}_2 \\ \vec{e}_3 & 0 & -\vec{e}_1 \\ -\vec{e}_2 & \vec{e}_1 & 0 \end{pmatrix} = \varepsilon^T\)

where \(E\) is the identity matrix, \(\varepsilon_{ijk}\) the tensor for permutation,

\(\sim\) is the tilde operator w.r.t. \(\varepsilon_{ijk}\)
4. Relation between (frame independent) vectors and matrices

<table>
<thead>
<tr>
<th>Vector (tensor) computations</th>
<th>Matrix calculations of coordinates of vectors w.r.t. basis axes ( \vec{e}_1, \vec{e}_2, \vec{e}_3 )</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Vector ( \vec{v} )</th>
<th>( \vec{v} = (v_i) = \begin{pmatrix} v_1 \ v_2 \ v_3 \end{pmatrix} ), ( i = 1,2,3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount (Length) ( v =</td>
<td>\vec{v}</td>
</tr>
<tr>
<td>Addition ( \vec{v} = \vec{a} + \vec{b} = \vec{b} + \vec{a} )</td>
<td>( \vec{v} = \vec{a} + \vec{b} = (a_i) + (b_i) = \begin{pmatrix} a_1 \ a_2 \ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \ b_2 \ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \ a_2 + b_2 \ a_3 + b_3 \end{pmatrix} )</td>
</tr>
<tr>
<td>Subtraction ( \vec{v} = \vec{a} - \vec{b} = -\vec{b} + \vec{a} )</td>
<td>( \vec{v} = \vec{a} - \vec{b} = (a_i) - (b_i) = \begin{pmatrix} a_1 \ a_2 \ a_3 \end{pmatrix} - \begin{pmatrix} b_1 \ b_2 \ b_3 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \ a_2 - b_2 \ a_3 - b_3 \end{pmatrix} )</td>
</tr>
<tr>
<td>Product scalar with vector ( \vec{v} = \lambda \vec{a} = \lambda (a_i) )</td>
<td>( \vec{v} = \lambda \vec{a} = \lambda (a_i) = \begin{pmatrix} \lambda a_1 \ \lambda a_2 \ \lambda a_3 \end{pmatrix} = \lambda \begin{pmatrix} e_{v1} \ e_{v2} \ e_{v3} \end{pmatrix} )</td>
</tr>
<tr>
<td>Scalar product ( \mu = \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} )</td>
<td>( \mu = \vec{a}^T \vec{b} = \vec{b}^T \vec{a} = a_1 b_1 + a_2 b_2 + a_3 b_3 )</td>
</tr>
<tr>
<td>Cross product ( \vec{v} = \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} )</td>
<td>( \vec{v} = \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} ) (also possible ( \vec{a} \equiv \hat{\vec{A}} ))</td>
</tr>
</tbody>
</table>

Note: \( \vec{a} \times \vec{a} = 0 \)

- Kinematic example \( \vec{v} = \vec{\omega} \times \vec{r} \)

- Static example \( \vec{M} = \vec{r} \times \vec{F} \)

| Note \( \vec{\omega} \vec{\omega} = \begin{pmatrix} -\omega_x z - \omega_z y & \omega_z \omega_y - \omega_y \omega_z \\ \omega_y \omega_z + \omega_z \omega_y & -\omega_z x - \omega_x z \\ \omega_z \omega_x - \omega_x \omega_z & -\omega_x \omega_y + \omega_y \omega_x \end{pmatrix} \) symm. |

\[ \begin{pmatrix} -\omega_x^2 & \omega_x \omega_y & \omega_x \omega_z \\ -\omega_y \omega_x & -\omega_y^2 & \omega_y \omega_z \\ -\omega_x \omega_z & -\omega_y \omega_z & -\omega_z^2 \end{pmatrix} \]
Dyadic product \( \vec{I} = \vec{a} \odot \vec{b} \)

= tensor type 2

\[
\begin{vmatrix}
I_{11} & I_{12} & I_{13} \\
I_{21} & I_{22} & I_{23} \\
I_{31} & I_{32} & I_{33}
\end{vmatrix} = \begin{pmatrix}
a_1 b_1 & a_1 b_2 & a_1 b_3 \\
a_2 b_1 & a_2 b_2 & a_2 b_3 \\
a_3 b_1 & a_3 b_2 & a_3 b_3
\end{pmatrix}
\]

5. Differentiation of Functions

Function \( a(\varphi(t)) \):

\[
\frac{da}{dt} = \dot{a} = \frac{\partial a}{\partial \varphi} \frac{d\varphi}{dt} = \frac{\partial a}{\partial \varphi} \dot{\varphi} = a_{\varphi}\dot{\varphi}
\]

Function \( a(\varphi(t), \gamma(t)) \):

\[
\frac{da}{dt} = \dot{a} = \frac{\partial a}{\partial \varphi} \dot{\varphi} + \frac{\partial a}{\partial \gamma} \dot{\gamma} = a_{\varphi}\dot{\varphi} + a_{\gamma}\dot{\gamma}
\]

6. Often used letters

\( K \) denotes a coordinate system or frame

\( I \) inertial frame

\( B \) body fixed frame

\( R \) reference frame

\( \vec{e}_i \) basis vectors, \( i = x, y, z \) or \( i, 2, 3 \); where unit vectors \( |\vec{e}_i| = 1 \)

\( x, y, z \) frame directions of \( K \)

\( X, Y, Z \) frame directions of inertial frame \( I \)

\( s, v, a \) values for position, velocity and acceleration

\( \alpha, \beta, \gamma, \varphi, \psi, \delta, \theta \) values for angle

\( \omega, \alpha \) angular velocity, angular acceleration

\( k_r = (k_{rx}, k_{ry}, k_{rz})^T \) coordinates of a vector w.r.t. frame \( k \), no index denotes inertial frame 0, 1, or \( I \).

\( A^{IB} \) 3 \( \times \) 3 orientation matrix of frame \( B \) w.r.t. \( I \): \( \vec{e}_I = A^{IB} \vec{e}_B \), or \( I \vec{v} = A^{IB} B \vec{v} \)

2D planar motion

3D spatial motion

2D planar motion

3D spatial motion

\( E \) identity matrix

\( A^T \) transposed Matrix \( A \); it leads to \( (A_{ij})^T = (A_{ji}) \)

\( A^{-1} \) inverse matrix \( A \); where \( A^{-1} A = E \), and \( E \) is the identity matrix
CAD  Computer Aided Design
FEM  Finite Element Method
MBS  Multibody System
AE   algebraic equations
DE   differential equations
DAE  differential algebraic equations
DOF  Degree of Freedom

$l, r, a, b, c, d, k, ..$  length
$e$                           eccentricity
$0, 1, 2, 3, ...$           numbers for links
$12, ...$                    number of a joint between link 1 and link 2
$A, B,$                      name of a point at links (name of a marker)
$A_0, B_0, ..$               name of a point at the ground
Sources and references

Recommendation References of this course

Elementary books of Mechanism Design and Analysis are (Kerle and Pittschellis 1998) or (Kerle, Pittschellis et al. 2007). An American bible is (Erdman and Sandor 1991).

Especially for German students (Jayendran 2006) and (Flack and Möllerke 1999) are proposed.

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### 3.3.2 Dead point construction of a crank-rocker mechanism via ALT

### 3.3.3 Find a crank-rocker mechanism for given function $\psi(\phi)$

### 3.3.4 A graphical method to find a crank-rocker for given function $\psi(\phi)$

### 3.4 Example Paths of Coupling Points

### 3.4.1 Find a crank-rocker mechanism for a given path of a coupler point

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### 3.5 Example Paths of a Specific Link

### 3.5.1 Find a 4-bar mechanism for a given path of a link

### 3.6 Example Cam Motions

### 3.7 Summary of Mechanism Design

### 4 Kinematics of Mechanisms

#### 4.1 Cartesian Coordinate Frames

##### 4.1.1 Definitions

##### 4.1.2 Transformation matrix and rotation coordinates

##### 4.1.3 Properties of a transformation matrix

##### 4.1.4 Calculation of angles of a transformation matrix

##### 4.1.5 Usage of frames to describe the motion of a body

#### 4.2 Velocity and Acceleration of a Body

#### 4.3 Constrained Planar Motions of a Body

##### 4.3.1 Body motion due to a prismatic joint

##### 4.3.2 Body motion due to a revolute joint

##### 4.3.3 Motion of a Rolling Cylindrical Body

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#### 5.1 Introduction

#### 5.2 Newton-Euler's Equations

#### 5.3 Jourdain's Principle
1 Introduction

1.1 What is a Mechanism?

A mechanism is a mechanical system which transfers motions or energies from the input side to the output side.

A Mechanism is an assemble of links (bars, bodies) which are connected by joints and force elements such as springs, dampers, actuators a.o.

Joints constrain the body motion and access their kinematics, force elements perform the body dynamics.

Example 1: Mechanism of a lighter – for transfer of energy

a) real construction

4-bar mechanism (4-bar linkage)
with 4 links (#1 = ground link)
with 4 joints (hinges at A₀, A, B, B₀)
with a spring and borders,
The system has one degree of freedom (DOF)

b) mechanism scheme

Example 2: Film transport of camera (Volmer 1989) – for transfer of motion

a) real construction

b) mechanism scheme
**Example 3:** A one-cylinder engine – for transfer of energy

**Exercise:** Discuss the mechanisms above w.r.t. joints and force elements. Please give other examples of mechanisms.
1.2 Classification of Mechanisms

We use mechanisms with different kind of motions, but often planar mechanisms are applied due to the simple kind of joints such as hinges and sliders. Herein, all motions are in the plane. We call mechanisms whose axes are intersect in one point spherical mechanisms. All others are denoted spatial mechanisms.

A second classification of mechanisms is the type of transfer function which is referred to them.

Tables 1-1 and 1-2 from (Volmer 1989) show this behaviour.

In general, mechanisms are designed in the sense that no deformations appear in the links. We talk about rigid bodies. In this course, all examples are considered to be rigid. Nevertheless, all arms and links of a mechanism will be deformed due to loads and in the case of high precision machines. Then these deformations have be considered in simulations. An exaggerated example is shown in Fig. 1-4. The links are so flexible that they bend due to the gravity force.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>spatial mechanism</td>
<td>all axes are arbitrary</td>
<td><img src="image1.png" alt="Example of a mechanism with all axes arbitrary" /></td>
</tr>
<tr>
<td>spherical mechanism</td>
<td>all axes are crossing at one point</td>
<td><img src="image2.png" alt="Example of a mechanism with all axes crossing at one point" /></td>
</tr>
<tr>
<td>planar mechanism</td>
<td>all axes are parallel</td>
<td><img src="image3.png" alt="Example of a mechanism with all axes parallel" /></td>
</tr>
</tbody>
</table>

Table 1-1: Mechanisms with different motions in space.

![Graph](image4.png)

Fig. 1-4: Example of a slider-crank mechanism with flexible crank and coupler (Schwertassek and Wallrapp 1999).
### Table 1-2: Mechanisms classification w.r.t. the transfer function.

Note: Gear mechanisms with non-circular wheels are also possible, see section 1.4.

A third classification of mechanisms is related to the applications. Many machines are in use in manufacturing and assembly processes. Referring to (VDI-2860 1990) Fig. 1-5 gives an overview; Fig. 1-6 shows some examples.
1.3 Elementary Mechanisms

The lowest elementary mechanism is a couple of links connected by a joint like a hinge: the 2-bar-linkage.

The following possible mechanisms are a combination of two 2-bar-linkages leading a 4-bar-linkage having 4 linkages and 4 joints. We get several elementary mechanisms:

- the crank-rocker mechanism

- slider-crank mechanism with 4 links, 3 hinges and 1 slider

- crank-shaper mechanism with 4 links, 3 hinges and 1 slider

- elliptic-trammel mechanism with 4 links, 2 hinges and 2 slider

All other mechanisms are extensions of these elementary mechanisms added by 2-bar-linkages.
1.4 Mechanisms with Specific Functions

This section shows some mechanisms having specific transfer functions.

Pantograph (transfer ratio $l = \text{line OC with respect to line OA}$)

A crane with a straight line motion of the path tracer point C, realised with a 4-bar mechanism

Gear wheel pairs with non-circular wheels having a non-linear transfer function
Handling machinery:
"Find all the link's lengths that are needed to take a box from the right side, turn it by 90° and put it down at the left side."

Gripper: "Find the linkages for the given input and output"

Pressure machinery

Micro gripper produced by Silicon
Sewing machine

Control unit of helicopter rotor blades (43 links, 4 DOFs)

Satellite with flexible yoke and 6 flexible panels (Wiedemann 1999).
Surgical tools using for dilating valves.

Surgical tools for the Minimal-Invasive-Surgery

Prosthetic knee mechanism (sketches for different walking positions)

The human body modelled as a mechanical system, see the course Biomechanics
1.5 Methods for the Analysis and Design of Mechanisms

We distinguish between Design (or Synthesis) and Analysis.

In the design, there is a idea of desired motion or energy flow and we want to find a machine which realise this idea. This question is often a significant process of engineering.

Second, there is a machine in form of a real system, a physical scaled model or a drawn model and we want to know, how does it works or what facts of motion it has. These process of engineering is called analysis of a machine. Tab. 1-3 summarises these statements. An overview of related programs is given in Tab. 1-4.

<table>
<thead>
<tr>
<th>Given</th>
<th>Wanted</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design or Synthesis</td>
<td>Motion of a path tracer point or a plane due to a given input motion</td>
<td>Required mechanism with length and angles of all links, as well as type of joints.</td>
</tr>
<tr>
<td>Analysis of Kinematics</td>
<td>Mechanism with motion of the input link</td>
<td>Rigid body motion of all other links and path tracer points, transfer function.</td>
</tr>
<tr>
<td>Analysis of Dynamics</td>
<td>Mechanism with motion of the input link as well as loads at all links</td>
<td>Required input force or torque, force and torques at joints (constraint forces) in addition to the motion of all other links and path tracer points, transfer function.</td>
</tr>
<tr>
<td>Analysis of Deformation</td>
<td>Mechanism with motion of the input link as well as loads at all links</td>
<td>Deformations, stress and strain of selected links during motion</td>
</tr>
</tbody>
</table>

Table 1-3: Methods in analysis and design of mechanisms
Specific Mechanism Programs:

Program *Approx for Windows* (Stauchmann 2002);
see authors home page [http://www.fh-muenchen.de/fb06/professoren/wallrapp/d_wallrapp_o.html](http://www.fh-muenchen.de/fb06/professoren/wallrapp/d_wallrapp_o.html)

General Purpose Programs

<table>
<thead>
<tr>
<th>Topic</th>
<th>CAD</th>
<th>FEM</th>
<th>MBS</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
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</tr>
</tbody>
</table>

Used for

Design of system, set-up data for geometry and material

Computation of strains and stresses due to loads

Computation of the nonlinear kinematics and dynamics of mechanical systems with rigid bodies

Options

analysis of kinematics, possibilities of synthesis FE-net generation

nonlinear kinematics and dynamics, Preparation of data for MBS

add flexible bodies, stress evaluation

Programs

Catia, Euklid
Pro-Engineer, AutoCAD, Solid Edge, Solid Works

ANSYS, ABAQUS, MARC, Nastran

ADAMS, DADS, SIMPACK, WorkingModel ReCurDyn

see (Brebbia 1982), (Schiehlen 1993), (Kortüm, Sharp et al. 1993)

Table 1-4: Programs for the Analysis and Design of Mechanisms